NECESSARY AND SUFFICIENT CONDITION OF MORSE-SARD THEOREM FOR REAL VALUED FUNCTIONS

AZAT AINOULINE

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Abstract

Necessary and sufficient condition is given for a set $A \subset \mathbb{R}^1$ to be a subset of the critical values set for a C^k function $f : \mathbb{R}^m \to \mathbb{R}^1$.

Introduction

The well known Morse Theorem [4] states that the critical values set $C_v f$ of a C^k map $f : \mathbb{R}^m \to \mathbb{R}^1$ is of measure zero in $\mathbb{R}^1$ if $k \geqslant m$, where $C_v f = f(C_p f)$ and $C_p f = \{x \in \mathbb{R}^m; rankDf = 0\}$ is the critical points set of f. Some generalizations of the necessary condition were made by a number of scientists. But the question is still open: whether a given measure zero set $A \subset R^1$ can be a set of critical values of some C^k function $f : \mathbb{R}^m \to \mathbb{R}^1$? In other words, what is necessary and sufficient condition of Morse theorem ? A successful approach to describe the critical values sets was made by Yomdin [7],[8], Bates and Norton [6] using a notion that in this paper we call 0_k-sets:

Definition 0.1 Let A be a compact subset of $\mathbb{R}$. We define a countable set $Z(A) = \{(\alpha, \beta) \subseteq \mathbb{R} \backslash A \; ; \; \alpha, \beta \in A, \; \alpha < \beta\}$. We call A a **set of k-degree,** $k > 0$, if the series $\sum_{z \in Z(A)} |z|^{\frac{1}{k}}$ converges, and designate A as $\mathbf{0_k}$**-set** if A is set of k-degree with measure 0.

The following theorem was proven by Bates and Norton:
Theorem ([6]). *A compact set $B = C_v f$ for some function $f \in C^k(\mathbb{R}, \mathbb{R})$ with compactly supported derivative if and only if B is 0_k-set.*

And according to [6], results equivalent to the theorem above were obtained by Yomdin in his unpublished paper [8], where he made a conjecture equivalent to the following:
Conjecture. *For $k \geqslant n$, a compact set $B = C_v f$ for some function $f \in C^k(\mathbb{R}^n, \mathbb{R})$ with compactly supported derivative if and only if B is $0_{k/n}$-set.*

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In this paper author gives necessary and sufficient condition for a set $A \subset \mathbb{R}^1$ to be a subset of the critical value set of a $C^{<k+\lambda}$ function $f : \mathbb{R}^m \to \mathbb{R}^1$.

Main Theorem. *A set $A \subseteq C_v f$ for some $C^{<k+\lambda}$ function $f : \mathbb{R}^m \to \mathbb{R}^1$, $m \leqslant k$, if and only if A is subset of a σ–$0_{<\frac{k+\lambda}{m}}$ set.*

And as a corollary, the necessary and sufficient condition is given for a set $A \subset \mathbb{R}^n$ to be subset of the image of the critical points set of rank zero of a $C^{<k+\lambda}$ function $f : \mathbb{R}^m \to \mathbb{R}^n$.

Corollary. *A set $A \subseteq C_v f$ for some $C^{<k+\lambda}$ function $f : \mathbb{R}^m \to \mathbb{R}^n$, $m \leqslant k$, if and only if its projection on any axis in $\mathbb{R}^n$ is a subset of a $\sigma - 0_{<\frac{k+\lambda}{m}}$ set,*

where we designate :
• for $k \in [1, \infty)$ a compact set $A \subseteq \mathbb{R}$ as $0_{<k}$-set if A is 0_t-set for every positive real $t < k$,
• A as a $\boldsymbol{\sigma - 0_k}$ **set** ($\boldsymbol{\sigma - 0_{<k}}$ **set**) if $A = \bigcup_{i\in\mathbb{N}} A_i$, where A_i is 0_k-set ($0_{<k}$-set) $\forall\ i \in \mathbb{N}$.

1 Definitions and Preliminary Lemmas

For $k \in \mathbb{N}$, a map $f : \mathbb{R}^n \to \mathbb{R}^m$ is of class C^k if it possesses continuous derivatives of all orders $\leqslant k$. For $\lambda \in [0, 1)$ f belongs to $C^{k+\lambda}$ provided $f \in C^k$ and the k^{th} derivative $D^k f$ satisfies a local Hölder condition: For each $x_0 \in \mathbb{R}^n$, there exists a neighborhood U of x_0 and a constant M such that

$$\|D^k f(x) - D^k f(y)\| \leqslant M|x - y|^\lambda$$

for all $x, y \in U$. If $f \in C^t$ for all $t < k + \lambda$ we write $f \in C^{<k+\lambda}$.
Generalized Morse Theorem.
Let $k, m \in \mathbb{N}$, and $\beta \in [0, 1)$. Let A be a subset of $\mathbb{R}^m$.
a)(see Norton [5]) *There exist subsets G_i $(i = 0, 1, 2 \dots)$ of A with $A = \bigcup_{i=1}^{\infty} G_i$ and any $f \in C^{k+\lambda}(\mathbb{R}^m, \mathbb{R})$ critical on A satisfies for each i : $|f(x) - f(y)| \leqslant M_i|x - y|^{k+\lambda}$ for all $x, y \in G_i$ and some $M_i > 0$.*
b) There exist subsets A_i $(i = 0, 1, 2 \dots)$ of A with $A = \bigcup_{i=1}^{\infty} A_i$ and any $f \in C^{<k+\lambda}(\mathbb{R}^m, \mathbb{R})$ critical on A satisfies for each i : $|f(x) - f(y)| \leqslant N_i|x - y|^{k+\lambda}$ for all $x, y \in A_i$ and some $N_i > 0$.

Proof.
a)See [5].
b) Similar to a). □

Definition 1.1 For $m, n \in \mathbb{N}$, $k \in \mathbb{R}$ a function $\psi : B \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is $\boldsymbol{D^k}$**-function** if $\exists K > 0$: $\forall b, b' \in B$ $|\psi(b) - \psi(b')|^k \leqslant K|b - b'|$.

We can rewrite the **Generalized Morse Theorem** in terms of D^k-functions:

Corollary 1.1 *Let $k, m \in \mathbb{N}$, and $\beta \in [0,1)$. Let A be a subset of $\mathbb{R}^m$.*
a) There are subsets G_i $(i = 0,1,2\dots)$ of A with $A = \bigcup_{i=1}^{\infty} G_i$ and for any $f \in C^{k+\lambda}(\mathbb{R}^m, \mathbb{R})$ critical on A, $f \restriction G_i$ is a $D^{\frac{1}{k+\lambda}}$-function for each i.
b) There are subsets A_i $(i = 0,1,2\dots)$ of A with $A = \bigcup_{i=1}^{\infty} A_i$ and for any $f \in C^{k+\lambda}(\mathbb{R}^m, \mathbb{R})$ critical on A, $f \restriction A_i$ is a D^{β}-function for each i, and every $\beta > \frac{1}{k+\lambda}$.

Properties of D^k-functions:

1) Extension on closure property. If $f : A \subseteq \mathbb{R}^m \overset{D^k}{\to} \mathbb{R}^n$ for some $k > 0$, and $\overline{A}$ is the closure of A, then there exists and unique C^0 function $\overline{f} : \overline{A} \subseteq \mathbb{R}^m \to \mathbb{R}^n$ such that $\overline{f} \restriction A = f$. And $\overline{f}$ is a D^k function.

Proof. $\forall a \in \overline{A}$ let us define $\overline{f}(a) = \lim\limits_{x \in A, x \to a} f(x)$, existence of the limit and the properties of $\overline{f}$ easy follow from the fact that $f \in D^k$.

2) Composition property. If $g \in D^k$ and $f \in D^p$, then $g \circ f \in D^{kp}$.

3) Subsets property. If $f : A \subseteq \mathbb{R}^m \overset{D^k}{\to} \mathbb{R}^m$ for some $k > 0$, then $f \restriction B \in D^k$ for any $B \subseteq A$.

Now we define a set $K_0^n = \{Q_{i_0},\ i_0 \in \mathbb{N}\}$, where every Q_{i_0} is a closed cube in $\mathbb{R}^n$ with side length 1 and every coordinate of any vertex of Q_{i_0} is an integer.
In general, having constructed the cubes of K_{s-1}^n, divide each $Q_{i_0,i_1,i_2,\dots,i_{s-1}} \in K_{s-1}^n$ into 2^n closed cubes of side $\frac{1}{2^s}$, and let K_s^n be the set of all those cubes. More precisely we will write
$K_s^n = \{Q_{i_0,i_1,i_2,\dots,i_{s-1},i_s}\ ;\ Q_{i_0,i_1,i_2,\dots,i_{s-1},i_s} \subseteq Q_{i_0,i_1,i_2,\dots i_{s-1}} \in K_{s-1}^n, 1 \leqslant i_s \leqslant 2^n\}$.
We use here a family of continuous space filling curves $f_n :\ [0,1] \overset{onto}{\longrightarrow} [0,1]^n$, $n \in \mathbb{N}$ with special properties:

$$\text{if } \alpha \subseteq [0,1] \text{ and } \exists s \in \mathbb{N} : \alpha \in K_{n\cdot s}^1 \implies f_n(\alpha) \subseteq \delta \text{ for some } \delta \in K_s^n \tag{1.1}$$
$$\text{if } \delta \subseteq [0,1]^n \text{ and } \exists s \in \mathbb{N} : \delta \in K_s^n \implies f_n^{-1}(int(\delta)) \subseteq \alpha \text{ for some } \alpha \in K_{n\cdot s}^1 \tag{1.2}$$

where $int(\delta)$ is the set of interior points of δ.

Definition 1.2 We call a function $f_n :\ [0,1] \overset{onto}{\to} [0,1]^n$ with the properties (1.1),(1.2) **cubes preserving**.

Theorem 1.1 ([1]) *For every $n \in \mathbb{N}$ there exists a continuous cubes preserving function $f_n : [0,1] \overset{onto}{\longrightarrow} [0,1]^n$.*

Lemma 1.1 *Any continuous space-filling cubes preserving function $f_n : [0,1] \overset{\text{onto}}{\to} [0,1]^n$ is a D^n-function.*

Proof. Let $a, b \in [0,1]$, $a < b$, then there exists $s_0 \in \mathbb{N}$ such that $\frac{1}{2^{n(s_0+1)}} \leqslant b - a \leqslant \frac{1}{2^{ns_0}}$. Then $[a,b] \subseteq \alpha' \cup \alpha''$ for some $\alpha', \alpha'' \in K_{ns_0}^1$, such that $\alpha' \cap \alpha'' \neq \emptyset$. From the

definition of cubes preserving function it follows that $f_n(\alpha') \subseteq \delta'$, $f_n(\alpha'') \subseteq \delta''$ for some $\delta', \delta'' \in K^n_{s_0}$ with $\delta' \cap \delta'' \neq \emptyset$ because $\alpha' \cap \alpha'' \neq \emptyset$. Now using $\operatorname{diam}(\delta' \cup \delta'') \leqslant 2\sqrt{n} \cdot \frac{1}{2^{s_0}}$, we get: $|f_n(b) - f_n(a)| \leqslant \operatorname{diam}(f_n(\alpha' \cup \alpha'')) \leqslant \operatorname{diam}(\delta' \cup \delta'') \leqslant 2\sqrt{n}\frac{1}{2^{s_0}}$ or

$$|f_n(b) - f_n(a)|^n \leqslant (2\sqrt{n}\frac{1}{2^{s_0}})^n = 2^{2n} n^{\frac{n}{2}} \frac{1}{2^{n(s_0+1)}} \leqslant K|b-a| \tag{1.3}$$

where $K = 2^{2n} \cdot n^{\frac{n}{2}}$. In case $a = b$ we have

$$|f_n(b) - f_n(a)|^n = 0 \leqslant K \cdot 0 = K \cdot |b-a|. \tag{1.4}$$

From (1.3) and (1.4) it follows that f_n is D^n-function on $[0,1]$. $\square$

Lemma 1.2 (see Corollary from Lemma 2 in [2] and also combinational lemma in [6]) *Let $f : [a,b] \to \mathbb{R}^1$ be continuous with $A \subseteq [a,b]$ compact and $B := f(A)$. There exists an injective function $\gamma : Z(B) \to Z(A)$ such that for each $z \in Z(B)$ with (say) $\gamma(z) = (x, x')$, one has $z \subseteq (f(x), f(x'))$.*

Lemma 1.3 *Let $f : [a,b] \subseteq \mathbb{R}^1 \to \mathbb{R}^1$ be a continuous function such that $f \restriction A$ is D^k- function for some closed $A \subseteq [a,b]$ and $k > 0$, then $f(A)$ is $\frac{1}{k}$-degree set.*

Proof. By the Lemma 1.2, there exists an injective function $\gamma : Z(f(A)) \to Z(A)$ such that for each $z \in Z(f(A))$ with (say) $\gamma(z) = (x, x') \in Z(A)$, one has $z \subseteq (f(x), f(x'))$ then $|z|^k \leqslant |f(x) - f(x')|^k$ for each $z \in Z(f(A))$ and $x, x' \in A$ (Note: $(x, x') \in Z(A)$ means that $x, x' \in A$). On the other hand for any $x, x' \in A$ $|f(x) - f(x')|^k \leqslant K|x - x'|$ for some $K > 0$ because $f \restriction A \in D^k$. Hence for each $z \in Z(f(A))$: $|z|^k \leqslant |f(x) - f(x')|^k \leqslant K|x - x'| = K|\gamma(z)|$, where $\gamma(z) = (x, x') \in Z(A)$. And consequently $\sum_{z \in Z(f(A))} |z|^k \leqslant K \sum_{z \in Z(f(A))} |\gamma(z)|$, but $\sum_{z \in Z(f(A))} |\gamma(z)| \leqslant \sum_{z \in Z(A)} |z|$ (because the function γ is injective), then $\sum_{z \in Z(f(A))} |z|^k \leqslant K \sum_{z \in Z(A)} |z| \leqslant K|b-a| < \infty$. So that the series $\sum_{z \in Z(f(A))} |z|^k$ converges and by the Definition 0.1 $f(A)$ is $\frac{1}{k}$-degree set. $\square$

2 Proof of necessary condition of Main Theorem

Theorem 2.1 *.Let $n \leqslant k \in \mathbb{N}$, $\lambda \in [0,1)$*
a) If $f \in C^{k+\lambda}(\mathbb{R}^n, \mathbb{R}^1)$, then $C_v f$ is a σ–$0_{\frac{k+\lambda}{n}}$ set .
b) If $F \in C^{<k+\lambda}(\mathbb{R}^n, \mathbb{R}^1)$, then $C_v F$ is a σ–$0_{<\frac{k+\lambda}{n}}$ set .

Proof.
a) By the Corollary 1.1, there exist G_i $(i = 0, 1, 2 \dots)$ such that for each i $f \restriction G_i$ is $D^{\frac{1}{k+\lambda}}$-function. Then for each i $f \restriction \overline{G}_i$ is $D^{\frac{1}{k+\lambda}}$ by the "*Extension on closure property of D^k-functions*" and $C_p f = \cup_{i=1}^{\infty} \overline{G}_i$ becouse $C_p f$ is closed .

We may suppose without loss of generality that for each i G_i and, therefore, $\overline{G}_i$, is contained in some closed cube $C_i \subseteq \mathbb{R}^n$ of side length 1 (because otherwise G_i can be represented as countable union of sets contained in such cubes).

Let for each $i \in \mathbb{N}$ $g_i : [0,1] \stackrel{onto}{\rightarrow} C_i$ be a continuous D^n function that exists by the Theorem 1.1 and Lemma 1.1.Then on the closed set $g_i^{-1}(\overline{G}_i)$ $g_i \restriction (g_i^{-1}(\overline{G}_i))$ is D^n- function by the "*Subsets property of D^k-functions*". And by the "*Composition property of D^k-functions*" $f \circ g_i \restriction (g_i^{-1}(\overline{G}_i))$ is a $D^{\frac{n}{k+\lambda}}$-function . So that by the Lemma 1.3 and by the fact that $f \circ g_i$ is continuous on $[0,1]$ as a composition of continuous functions, it follows that the closed set $f(g_i(g_i^{-1}(\overline{G}_i)))$ is of $\frac{k+\lambda}{n}-$degree set, and therefore $f(\overline{G}_i)$ is of $\frac{k+\lambda}{n}$-degree set.

By the Morse Theorem [4] $f(C_p f) = C_v f$ is a measure zero set.And becouse for each i $\overline{G}_i \subseteq C_p f$ it follows that $f(\overline{G}_i)$ is $0_{\frac{k+\lambda}{n}}$-set , and consequently $f(C_p f) = C_v f$ is a $\sigma - 0_{\frac{k+\lambda}{n}}$ set (recall that $C_p f = \bigcup_{i=0}^{\infty} \overline{G}_i$).

b) Similar to a). The Theorem 2.1 is proven, and thereby the prove of the necessary condition of the Main Theorem has been finished.

3 Proof of sufficient condition of Main Theorem

Theorem 3.1 *If A is a $\sigma - 0_{<s}$-set $(1 < s \in \mathbb{R})$, then for every $n \in \mathbb{N}$ $A \subseteq C_v F$ for some $C^{<sn}$ function $F : \mathbb{R}^n \to \mathbb{R}^1$.*

Proof. First let us fix such s, n and prove that for any $0_{<s}$ set B there exists a $C^{<sn}$ function $f : [0,1]^n \to \mathbb{R}$ with $B \subseteq C_v F$. Since B is an $0_{<s}$-set, then the series $\sum_{z_n \in Z(B)} |z_n|^{1/t}$ converges for any positive $t < s$. The case $B = \emptyset$ is trivial, so we suppose $B \neq \emptyset$.

Let $P \geqslant 1$ denote the greatest integer less than sn and $\{s_m,\ m \in \mathbb{N}\}$ be a non-decreasing sequence such that $(P + sn)/2 < s_m < sn$, $\lim_{m\to\infty} s_m = sn$ and $\sum_{z_m \in Z(B)} |z_m|^{1/s_m}$ is convergent. The existence of such sequence $\{s_m\}$ can be seen from the fact that $\forall i \in \mathbb{N}$, $\forall t \in \mathbb{R}^+$, $t < sn$, there exists $m_i \in \mathbb{N}$ such that $\sum_{m > m_i, z_m \in Z(B)} |z_m|^{1/t} < \frac{1}{2^i}$, and $\sum_{i=1}^{\infty} \frac{1}{2^i} = 2$.

For any closed $B^* \subseteq B$ let us designate the sum $\sum_{z_m \in Z(B^*)} |z_m|^{1/s_m} = G(B^*)$. Further we construct the function f following the scheme of Bates ([3],sec.1.4) and replacing definitions of P, $R(a_k)$ and $\sigma_t(k)$ in his construction.

3.1 Construction of f

For $\beta \in (0, 1/2)$, we can first define the following method for constructing 2^n cubes within any cube in $\mathbb{R}^n$:
Supposing $Q \subset \mathbb{R}^n$ is a cube of side $L > 0$ defined by the inequalities $|x_i - c_i| \leqslant L/2$, we specify 2^n subcubes within Q with the inequalities $|x_i - c_i \pm L/4| \leqslant \beta L/2$. Note that each subcube is separated from all other subcubes and the boundary of Q by a distance $\geqslant L(1/4 - \beta/2)$.

3.1.1 A Cantor set in $\mathbb{R}^n$.

Let $Q_0 \subset \mathbb{R}^n$ be the cube defined by $|x_i| \leqslant 1/2$. For $i \in \mathbb{Z}^+$ let $\beta_i = (1/2) \cdot e^{-1/i}$ and set $\pi_k = (16k)^{-1} \cdot \prod_{i=1}^{k} \beta_i$. For $k \in \mathbb{Z}^+$ the index a_k always denotes a k-tuple

of numbers in $\{1, 2, 3, ..., 2^n\}$. We now define a system of subcubes in Q_0 as follows:

a. Let $\{Q(a_1)\}$ be the 2^n subcubes constructed in Q_0 by the method described above with $\alpha = \beta_1$.

b. Having defined $\{Q(a_k)\}$, construct 2^n subcubes within each $Q(a_k)$ according to the above process with $\alpha = \beta_{k+1}$ and label these subcubes $\{Q(a_k, i)\}$ for $i = 1, ..., 2^n$.

Evidently, the cube $Q(a_k)$ has side length $L_k = \prod_{i=1}^k \beta_i$; for integers $k \leqslant l$ the boundaries of distinct cubes $Q(a_k)$ and $Q(a'_l)$ are separated by a distance $\geqslant (1/4 - \beta_{k+1}/2)L_k \geqslant \pi_k$.

Let ζ denote the Cantor set defined by the cubes, i.e. the set of points in Q_0 contained in indefinitely many of subcubes constructed above.

3.1.2 Mapping of ζ.

Let $R_0 = B$. For each $k \in \mathbb{Z}^+$, choose decomposition of R_0 into 2^{nk} non-empty closed set $\{R(a_k)\}$ such that $G(R(a_k)) \leqslant M \cdot 2^{-nk}$ and $R(a_k) = \cup_{i=1}^{2^n} R(a_k, i)$. For each a_k, fix a point $r(a_k) \in R(a_k)$.

Define the map f on ζ by the requirement that for each index a_k, $f(\zeta \cap Q(a_k)) \subseteq R(a_k)$. The conditions imposed on the sets $R(a_k)$ then insure that $R_0 \subseteq f(\zeta)$.

3.1.3 Extension of f.

Consider the cube $Q = Q(a_k)$ and its subcubes $Q_i = Q(a_k, i)$. For each $i = 1, ..., 2^n$ choose a function $h_i : \mathbb{R}^n \to \mathbb{R}$ such that

(1) $h_i = 1$ on a neighborhood of Q_i;

(2) $Supp(h_i) \subset Int\ Q$, and $Supp(h_i) \cap Supp(h_j) = \emptyset$ whenever $i \neq j$.

In view of condition (2) and the distance between the Q_i, we can choose the h_i so that, for each $p \in \mathbb{Z}^+$,

(3) $\|D^p h_i\| \leqslant M_p(\pi_{k+1})^{-p}$.

Now we define the partial extension of f to the region $Q \setminus \bigcup Q_i$ by

$$f = r + \sum_{i=1}^{2^n} (r_i - r)h_i,$$

where $r = r(a_k)$, $r_i = r(a_k, i)$.

3.1.4 Smoothness of f.

For $t \in \mathbb{R}$ define $\sigma_t(k) = 2^{-(sn+t)k/2} \cdot (\pi_{k+1})^{-t}$. Since $\beta_k \to 1/2$ as $k \to \infty$, a simple estimation shows that π_k is bounded below by $M' \cdot 2^{-k} k^{-2}$. Consequently, for $t < sn \quad \sigma_t(k) \to 0$ as $k \to \infty$.

To determine the smoothness of f, we first observe that $(P + sn)/2n < s$, also if $k \geqslant 1$ is the largest integer such that $x, x' \in Q(a_k) \cap \zeta$, then $|x - x'| \geqslant \pi_{k+1}$, and

by the definition of f,

$$\begin{aligned}
&|f(x)-f(x')| \leqslant diam(R(a_k)) = \sum\nolimits_{z_m\in Z(R(a_k))} |z_m| \\
&\leqslant \left(\sum\nolimits_{z_m\in Z(R(a_k))} |z_m|^{2n/(P+sn)}\right)^{(P+sn)/2n} \leqslant (G(R(a_k)))^{(P+sn)/2n} \\
&\leqslant M'\sigma_P(k)\pi_{k+1} \leqslant M'\sigma_P(k)|x-x'|^P.
\end{aligned}$$

This implies that $f \restriction \zeta$ is continuous; by condition (1), it follows that the partial extension defined above comprise a continuous extension of $f \restriction \zeta$. Since condition (2) implies $\|D^p f\| = 0$ on boundaries $\partial Q(a_k)$ for all $k, p \in \mathbb{Z}^+$, it follows furthermore that f is C^∞ on $\mathbb{R}^n \setminus \zeta$.

Condition (3) above implies that on $Q(a_k) \setminus \bigcup Q(a_k, i)$

$$\|D^p f\| \leqslant M_p(\pi_{k+1})^{-p} \cdot diam(R(a_k)) \leqslant M'_p\sigma_p(k).$$

Consequently $D^p f \to 0$ on approach to ζ whenever $p \leqslant P$. By an application of the mean-value theorem to the inequality found for $f \restriction \zeta$ above, it follows that $D^p f = 0$ on ζ for $p \leqslant P$, and so f is C^P on $\mathbb{R}^n$.

Given $t \in (P, sn)$, we observe that if $x \in \zeta$ and $k \gg 1$ is again the largest integer such that $x, y \in Q(a_k)$, then the same argument used above shows that

$$\|D^P f(x) - D^P f(y)\|/|x-y|^{t-P} = \|D^P f(y)\|/|x-y|^{t-P} \leqslant M''_P\sigma_t(k+1).$$

Since $f \in C^\infty$ outside ζ, this inequality implies $f \in C^t$ throughout $\mathbb{R}^n$. Evidently, $rankDf = 0$ on ζ, and $B \subseteq C_v f$.

By construction, f is constant outside Q_0. Now for a $\sigma - O_{<s}$ set A, we can define $C^{<sn}$ function $F : \mathbb{R}^n \to \mathbb{R}^1$, such that $A \subseteq C_v F$:
We take $\{Q_0^i\}_{i\in\mathbb{N}}$ - a discrete family of cubes of side length 1 in $\mathbb{R}^n$ and a family of $C^{<sn}$ functions $\{f_i : Q_0^i \to \mathbb{R}^1 \ ; \ A_i \subseteq C_v f_i\}$. For each i we define $F = f_i \restriction Q_0^i$. And we can choose distances between cubes allowing F be C^∞ on $\mathbb{R}^n \setminus \cup Q_0^i$, and $C^{<sn}$ on $\mathbb{R}^n$. □

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